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ORIGINAL PAPER

Soliton solutions for time fractional coupled modified KdV equations using new coupled fractional reduced differential transform method

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Abstract In this paper, new Coupled Fractional Reduced Differential Transform has been implemented to obtain the soliton solutions of coupled time fractional modified KdV equations. This new method has been revealed by the author. The fractional derivatives are described in the Caputo sense. By using the present method, we can solve many linear and nonlinear coupled fractional differential equations. The results reveal that the proposed method is very effective and simple for obtaining approximate solutions of fractional coupled modified KdV equations. Numerical solutions are presented graphically to show the reliability and efficiency of the method. Solutions obtained by this new method have been also compared with Adomian decomposition method (ADM).

Keywords Coupled fractional reduced differential transform · Adomian decomposition method · Fractional coupled modified KdV equations · Caputo fractional derivative · Riemann–Liouville fractional derivative

1 Introduction

In the field of engineering, physics, other field of applied sciences many phenomena can be obtained very successfully by models using mathematical tools form of fractional calculus [1–8]. In the past decades, the fractional differential equations have been widely used in various fields of applied science and engineering. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, control theory, neutron point kinetic model, anomalous diffusion, vibration and control,

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continuous time random walk, Levy Statistics, Brownian motion, signal and image processing, relaxation, creep, chaos, fluid dynamics and material science are well described by differential equations of fractional order. Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. For that reason, we need a reliable and efficient technique for the solution of fractional differential equations. An immense effort has been expended over the last many years to find robust and efficient numerical and analytical methods for solving such fractional differential equations. In the present analysis, a new approximate numerical technique, Coupled Fractional Reduced Differential transform method (CFRDTM), has been applied which is applicable for coupled fractional differential equations. The new method is a very powerful solver for linear and non-linear coupled fractional differential equations. It is relatively a new approach to provide the solution very efficiently and accurately.

In this paper, coupled modified KdV equations [9,10], of time fractional order, have been considered. The paper is organized as follows: in Sect. 2, a brief review of the theory of fractional calculus has been provided for the precise purpose of this paper. In Sect. 3, the Coupled Fractional Reduced Differential Transform method has been analyzed in details. In Sect. 4, CFRDTM has been applied to determine the approximate solutions for the coupled time fractional modified KdV equations. The obtained results show the efficiency and simplicity of the proposed method. Finally, conclusions are presented.

2 Brief description of fractional calculus

The fractional calculus was first anticipated by Leibnitz, was one of the founders of standard calculus, in a letter written in 1695. This calculus involves different definitions of the fractional operators as well as the Riemann–Liouville fractional derivative, Caputo derivative, Riesz derivative and Grunwald–Letnikov fractional derivative [1]. The fractional calculus has gained considerable importance during the past decades mainly due to its applications in diverse fields of science and engineering. For the purpose of this paper the Caputo's definition of fractional derivative will be used, taking the advantage of Caputo's approach that the initial conditions for fractional differential equations with Caputo's derivatives take on the traditional form as for integer-order differential equations.

2.1 Definition-Riemann–Liouville integral

The most frequently encountered definition of an integral of fractional order is the Riemann–Liouville integral [1], in which the fractional integral of order $\alpha(>0)$ is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \; t > 0, \; \alpha \in \boldsymbol{R}^+ \tag{2.1}$$

where $\boldsymbol{R}^+$ is the set of positive real numbers.

2.2 Definition-Caputo fractional derivative

The fractional derivative, introduced by Caputo [11,12] in the late sixties, is called Caputo Fractional Derivative. The fractional derivative of $f(t)$ in the Caputo sense is defined by

$$D_t^{\alpha} f(t) = J^{m-\alpha} D^m f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{(m-\alpha-1)} \frac{d^m f(\tau)}{d\tau^m} d\tau, & if\ m-1 < \alpha < m, m \in \boldsymbol{N} \\ \frac{d^m f(t)}{dt^m}, & if\ \alpha = m, m \in \boldsymbol{N} \end{cases} \tag{2.2}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper only real and positive α will be considered.

For the Caputo's derivative we have

$$D^{\alpha} C = 0, \ (C \text{is a constant}) \tag{2.3}$$

$$D^{\alpha} t^{\beta} = \begin{cases} 0, & \beta \leq \alpha - 1 \\ \frac{\Gamma(\beta+1) t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, & \beta > \alpha - 1 \end{cases} \tag{2.4}$$

Similar to integer order differentiation Caputo's derivative is linear.

$$D^{\alpha}(\gamma f(t) + \delta g(t)) = \gamma D^{\alpha} f(t) + \delta D^{\alpha} g(t) \tag{2.5}$$

where γ and δ are constants, and satisfies so called Leibnitz's rule.

$$D^{\alpha}(g(t) f(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) D^{\alpha-k} f(t) \tag{2.6}$$

If $f(\tau)$ is continuous in $[0, t]$ and $g(\tau)$ has $n+1$ continuous derivatives in $[0, t]$.

2.3 Lemma

If $m - 1 < \alpha < m, m \in \boldsymbol{N}$, then

$$D^{\alpha} J^{\alpha} f(t) = f(t) \tag{2.7}$$

and

$$J^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0+), t > 0 \tag{2.8}$$

2.4 Theorem

(Generalized Taylor's formula) [13] Suppose that $D_a^{k\alpha} f(t) \in C(a, b]$ for $k = 0, 1, \ldots, n+1$, where $0 < \alpha \leq 1$, we have

$$f(t) = \sum_{i=0}^{n} \frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left[D_a^{k\alpha} f(t)\right]_{t=a} + \Re_n^{\alpha}(t;a) \tag{2.9}$$

with $\Re_n^{\alpha}(t;a) = \frac{(t-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \left[D_a^{(n+1)\alpha} f(t)\right]_{t=\xi}$, $a \leq \xi \leq t, \forall t \in (a, b]$, where $D_a^{k\alpha} = D_a^{\alpha}.D_a^{\alpha}.D_a^{\alpha} \cdots D_a^{\alpha}$ (k times).

3 Coupled fractional reduced differential transform method (CFRDTM)

In order to introduce coupled fractional reduced differential transform, $U(h, k-h)$ is considered as the coupled fractional reduced differential transform of $u(x,t)$. If function $u(x,t)$ is analytic and differentiated continuously with respect to time t, then we define the fractional coupled reduced differential transform of $u(x,t)$ as

$$U(h, k-h) = \frac{1}{\Gamma(h\alpha + (k-h)\beta + 1)} \left[D_t^{(h\alpha+(k-h)\beta)} u(x,t)\right]_{t=0} \tag{3.1}$$

whereas the inverse transform of $U(h, k-h)$ is

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{k} U(h, k-h) t^{h\alpha+(k-h)\beta} \tag{3.2}$$

which is one of the solution of coupled fractional differential equations.

Theorem 1 *Suppose that $U(h, k-h)$, $V(h, k-h)$ and $W(h, k-h)$ are the Coupled Fractional Reduced Differential Transform of the functions $u(x,t)$, $v(x,t)$ and $w(x,t)$, respectively.*

i If $u(x,t) = f(x,t) \pm g(x,t)$ then $U(h, k-h) = F(h, k-h) \pm G(h, k-h)$.
ii If $u(x,t) = af(x,t)$, where $a \in \boldsymbol{R}$, then $U(h, k-h) = aF(h, k-h)$.
iii If $f(x,t) = u(x,t)v(x,t)$, then $F(h, k-h) = \sum_{l=0}^{h} \sum_{s=0}^{k-h} U(h-l, s) V(l, k-h-s)$.
iv If $f(x,t) = D_t^{\alpha} u(x,t)$, then

$$F(h, k-h) = \frac{\Gamma((h+1)\alpha + (k-h)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} U(h+1, k-h).$$

v If $f(x,t) = D_t^{\beta} v(x,t)$, then

$$F(h, k-h) = \frac{\Gamma(h\alpha + (k-h+1)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} V(h, k-h+1).$$

4 Soliton solutions for time fractional coupled modified KdV equations

Example 4.1 Consider the following time fractional coupled modified KdV equations [10]

$$D_t^{\alpha} u = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3u^2\frac{\partial u}{\partial x} + \frac{3}{2}\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial (uv)}{\partial x} - 3\frac{\partial u}{\partial x} \tag{4.1a}$$

$$D_t^{\beta} v = -\frac{\partial^3 v}{\partial x^3} - 3v\frac{\partial v}{\partial x} - 3\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 3u^2\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial x} \tag{4.1b}$$

where $t > 0, 0 < \alpha, \beta \leq 1$,
subject to the initial conditions

$$u(x,0) = \frac{1}{2} + \tanh(x) \tag{4.1c}$$

$$v(x,0) = 1 + \tanh(x) \tag{4.1d}$$

The exact solutions of Eqs. (4.1a) and (4.1b), for the special case where $\alpha = \beta = 1$, are given by

$$u(x,t) = \frac{1}{2} + \tanh(x+ct) \tag{4.2a}$$

$$v(x,t) = 1 + \tanh(x+ct) \tag{4.2b}$$

In order to assess the advantages and the accuracy of the CFRDTM for solving time fractional coupled modified KdV equations. Firstly, we derive the recursive formula from Eqs. (4.1a), (4.1b). Now, $U(h, k-h)$ and $V(h, k-h)$ are considered as the coupled fractional reduced differential transform of $u(x,t)$ and $v(x,t)$, respectively, where $u(x,t)$ and $v(x,t)$ are the solutions of coupled fractional differential equations. Here, $U(0,0) = u(x,0)$, $V(0,0) = v(x,0)$ are the given initial conditions. Without loss of generality, the following assumptions have taken

$$U(0, j) = 0, \quad j = 1, 2, 3, \cdots \text{ and } V(i, 0) = 0, \quad i = 1, 2, 3, \cdots$$

Applying CFRDTM to Eq. (4.1a), we obtain the following recursive formula

$$\begin{aligned}
&\frac{\Gamma((h+1)\alpha + (k-h)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} U(h+1, k-h) \\
&= \frac{1}{2}\frac{\partial^3}{\partial x^3} U(h, k-h) + \frac{3}{2}\frac{\partial^2}{\partial x^2} V(h, k-h) - 3\frac{\partial}{\partial x} U(h, k-h) \\
&\quad + 3\frac{\partial}{\partial x}\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(h-l, s) V(l, k-h-s)\right) \\
&\quad - 3\left(\sum_{r=0}^{h}\sum_{l=0}^{h-r}\sum_{s=0}^{k-h}\sum_{p=0}^{k-h-s} U(r, k-h-s-p) U(l, s) \frac{\partial}{\partial x} U(h-r-l, p)\right)
\end{aligned} \tag{4.3}$$

From the initial condition of Eq. (4.1c), we have

$$U(0,0) = u(x,0) \tag{4.4}$$

In the same manner, we can obtain the following recursive formula from Eq. (4.1b)

$$\begin{aligned}
&\frac{\Gamma(h\alpha+(k-h+1)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}V(h,k-h+1)\\
&= -\frac{\partial^3}{\partial x^3}V(h,k-h)+3\frac{\partial}{\partial x}V(h,k-h)\\
&\quad -3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}\frac{\partial}{\partial x}U(l,k-h-s)\frac{\partial}{\partial x}V(h-l,s)\right)\\
&\quad -3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}V(l,k-h-s)\frac{\partial}{\partial x}V(h-l,s)\right)\\
&\quad +3\left(\sum_{r=0}^{h}\sum_{l=0}^{h-r}\sum_{s=0}^{k-h}\sum_{p=0}^{k-h-s}U(r,k-h-s-p)U(l,s)\frac{\partial}{\partial x}U(h-r-l,p)\right)
\end{aligned} \tag{4.5}$$

From the initial condition of Eq. (4.1d), we have

$$V(0,0) = v(x,0) \tag{4.6}$$

According to CFRDTM, using recursive Eq. (4.3) with initial condition Eq. (4.4) and also using recursive scheme Eq. (4.5) with initial condition Eq. (4.6) simultaneously, we obtain

$$\begin{aligned}
U(1,0) &= -\frac{\operatorname{sech}^2(x)}{4\Gamma(1+\alpha)}\\
V(0,1) &= -\frac{\operatorname{sech}^2(x)}{4\Gamma(1+\beta)}\\
U(1,1) &= \frac{3\operatorname{sech}^2(x)\tanh(x)}{4\Gamma(1+\alpha+\beta)}\\
V(0,2) &= \frac{\operatorname{sech}^5(x)(9\cosh(x)-3\cosh(3x)+32\sinh(x)-4\sinh(3x))}{8\Gamma(1+2\beta)}\\
U(2,0) &= -\frac{7\operatorname{sech}^2(x)\tanh(x)}{8\Gamma(1+2\alpha)}\\
V(1,1) &= \frac{3\operatorname{sech}^5(x)(-12\cosh(x)+4\cosh(3x)-43\sinh(x)+5\sinh(3x))}{32\Gamma(1+\alpha+\beta)}
\end{aligned}$$

and so on.

The approximate solutions, obtained in the series form, are given by

$$
\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= \frac{1}{2} + \tanh(x) - \frac{t^{\alpha}\,\mathrm{sech}^2(x)}{4\Gamma(1+\alpha)} - \frac{7t^{2\alpha}\,\mathrm{sech}^2(x)\tanh(x)}{8\Gamma(1+2\alpha)} \\
&\quad + \frac{3t^{\alpha+\beta}\,\mathrm{sech}^2(x)\tanh(x)}{4\Gamma(1+\alpha+\beta)} + \cdots
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
v(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= 1 + \tanh(x) - \frac{t^{\beta}\,\mathrm{sech}^2(x)}{4\Gamma(1+\beta)} \\
&\quad + \frac{t^{2\beta}\,\mathrm{sech}^5(x)(9\cosh(x) - 3\cosh(3x) + 32\sinh(x) - 4\sinh(3x))}{8\Gamma(1+2\beta)} \\
&\quad + \frac{3t^{\alpha+\beta}\,\mathrm{sech}^5(x)(-12\cosh(x) + 4\cosh(3x) - 43\sinh(x) + 5\sinh(3x))}{32\Gamma(1+\alpha+\beta)} + \cdots
\end{aligned} \tag{4.8}
$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (4.7) becomes

$$
\begin{aligned}
u(x,t) &= \frac{1}{2} + \tanh(x) - \frac{t\,\mathrm{sech}^2(x)}{4} - \frac{t^2\,\mathrm{sech}^2(x)\tanh(x)}{16} \\
&\quad - \frac{t^3\,\mathrm{sech}^4(x)(-2+\cosh(2x))}{192} + \cdots
\end{aligned} \tag{4.9}
$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (4.8) becomes

$$
\begin{aligned}
v(x,t) &= 1 + \tanh(x) - \frac{t\,\mathrm{sech}^2(x)}{4} - \frac{t^2\,\mathrm{sech}^2(x)\tanh(x)}{16} \\
&\quad - \frac{t^3\,\mathrm{sech}^4(x)(-2+\cosh(2x))}{192} + \cdots
\end{aligned} \tag{4.10}
$$

The solutions in Eqs. (4.9) and (4.10) are exactly same as the Taylor series expansions of the exact solutions

$$
\begin{aligned}
u(x,t) &= \frac{1}{2} + \tanh\left(x - \frac{t}{4}\right) \\
&= \frac{1}{2} + \tanh(x) - \frac{t\,\mathrm{sech}^2(x)}{4} - \frac{t^2\,\mathrm{sech}^2(x)\tanh(x)}{16}
\end{aligned}
$$

$$-\frac{t^3\,\text{sech}^4(x)(-2+\cosh(2x))}{192}+\cdots \tag{4.11}$$

$$\begin{aligned} v(x,t) &= 1+\tanh\left(x-\frac{t}{4}\right)\\ &= 1+\tanh(x)-\frac{t\,\text{sech}^2(x)}{4}-\frac{t^2\,\text{sech}^2(x)\tanh(x)}{16}\\ &\quad -\frac{t^3\,\text{sech}^4(x)(-2+\cosh(2x))}{192}+\cdots \end{aligned} \tag{4.12}$$

In order to explore the efficiency and accuracy of the proposed method for the time fractional coupled modified KdV equations, the graphs have been drawn in Fig. 1a–d. The numerical solutions for Eqs. (4.9) and (4.10) for the special case where $\alpha = 1$ and $\beta = 1$ are shown in Fig. 1a, b. It can be observed from Fig. 1a–d that the solutions obtained by the proposed method coincide with the exact solution. In this case, we see that the soliton solutions are kink-types for both $u(x,t)$ and $v(x,t)$.

Example 4.2 Consider the following time fractional coupled modified KdV equations [9]

$$D_t^{\alpha} u = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3u^2\frac{\partial u}{\partial x} + \frac{3}{2}\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial (uv)}{\partial x} + 3\frac{\partial u}{\partial x} \tag{4.13a}$$

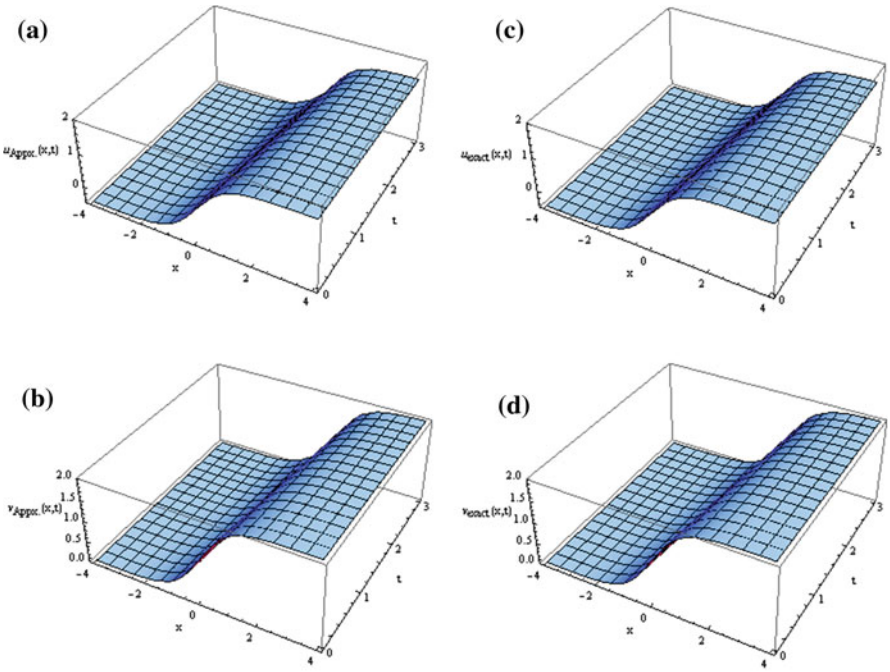

Fig. 1 The surfaces show **a** the numerical approximate solution of $u(x,t)$, **b** the numerical approximate solution of $v(x,t)$, **c** the exact solution of $u(x,t)$, and **d** the exact solution of $v(x,t)$ when $\alpha = 1$ and $\beta = 1$

$$D_t^{\beta} v = -\frac{\partial^3 v}{\partial x^3} - 3v\frac{\partial v}{\partial x} - 3\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 3u^2\frac{\partial v}{\partial x} - 3\frac{\partial v}{\partial x} \tag{4.13b}$$

where $t > 0, 0 < \alpha, \beta \leq 1$,
subject to the initial conditions

$$u(x, 0) = \tanh(x) \tag{4.13c}$$
$$v(x, 0) = 1 - 2\tanh^2(x) \tag{4.13d}$$

The exact solutions of Eqs. (4.13a) and (4.13b) obtained by Adomian decomposition method, for the special case where $\alpha = \beta = 1$, are given by

$$u(x, t) = \tanh(x - t) \tag{4.14a}$$
$$v(x, t) = 1 - 2\tanh^2(x - t) \tag{4.14b}$$

In order to assess the advantages and the accuracy of the CFRDTM for solving time fractional coupled modified KdV equations. Firstly, we derive the recursive formula from Eqs. (4.13a), (4.13b). Now, $U(h, k - h)$ and $V(h, k - h)$ are considered as the coupled fractional reduced differential transform of $u(x, t)$ and $v(x, t)$, respectively, where $u(x, t)$ and $v(x, t)$ are the solutions of coupled fractional differential equations. Here, $U(0, 0) = u(x, 0)$, $V(0, 0) = v(x, 0)$ are the given initial conditions. Without loss of generality, the following assumptions have taken

$$U(0, j) = 0, \quad j = 1, 2, 3, \cdots \text{ and } V(i, 0) = 0, \quad i = 1, 2, 3, \cdots .$$

Applying CFRDTM to Eq. (4.13a), we obtain the following recursive formula

$$\begin{aligned}
&\frac{\Gamma((h+1)\alpha + (k-h)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} U(h+1, k-h) \\
&= \frac{1}{2}\frac{\partial^3}{\partial x^3} U(h, k-h) + \frac{3}{2}\frac{\partial^2}{\partial x^2} V(h, k-h) \\
&\quad + 3\frac{\partial}{\partial x} U(h, k-h) + 3\frac{\partial}{\partial x}\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(h-l, s)V(l, k-h-s)\right) \\
&\quad - 3\left(\sum_{r=0}^{h}\sum_{l=0}^{h-r}\sum_{s=0}^{k-h}\sum_{p=0}^{k-h-s} U(r, k-h-s-p)U(l, s)\frac{\partial}{\partial x}U(h-r-l, p)\right)
\end{aligned} \tag{4.15}$$

From the initial condition of Eq. (4.13c), we have

$$U(0, 0) = u(x, 0) \tag{4.16}$$

In the same manner, we can obtain the following recursive formula from Eq. (4.13b)

$$\begin{aligned}
&\frac{\Gamma(h\alpha+(k-h+1)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}V(h,k-h+1)\\
&=-\frac{\partial^3}{\partial x^3}V(h,k-h)-3\frac{\partial}{\partial x}V(h,k-h)\\
&\quad-3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}\frac{\partial}{\partial x}U(l,k-h-s)\frac{\partial}{\partial x}V(h-l,s)\right)\\
&\quad-3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}V(l,k-h-s)\frac{\partial}{\partial x}V(h-l,s)\right)\\
&\quad+3\left(\sum_{r=0}^{h}\sum_{l=0}^{h-r}\sum_{s=0}^{k-h}\sum_{p=0}^{k-h-s}U(r,k-h-s-p)U(l,s)\frac{\partial}{\partial x}U(h-r-l,p)\right)
\end{aligned} \tag{4.17}$$

From the initial condition of Eq. (4.13d), we have

$$V(0,0)=v(x,0) \tag{4.18}$$

According to CFRDTM, using recursive Eq. (4.15) with initial condition Eq. (4.16) and also using recursive scheme Eq. (4.17) with initial condition Eq. (4.18) simultaneously, we obtain

$$\begin{aligned}
U(1,0)&=-\frac{\operatorname{sech}^2(x)}{\Gamma(1+\alpha)}\\
V(0,1)&=\frac{4\operatorname{sech}^2(x)\tanh(x)}{\Gamma(1+\beta)}\\
U(1,1)&=-\frac{24\operatorname{sech}^4(x)\tanh(x)}{\Gamma(1+\alpha+\beta)}\\
V(0,2)&=\frac{\operatorname{sech}^6(x)(21-26\cosh(2x)+\cosh(4x))}{\Gamma(1+2\beta)}\\
U(2,0)&=-\frac{(-23+\cosh(2x))\operatorname{sech}^4(x)\tanh(x)}{\Gamma(1+2\alpha)}\\
V(1,1)&=\frac{48\operatorname{sech}^4(x)\tanh^2(x)}{\Gamma(1+\alpha+\beta)}
\end{aligned}$$

and so on.

The approximate solutions, obtained in the series form, are given by

$$
\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= \tanh(x) - \frac{t^{\alpha}\,\mathrm{sech}^2(x)}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}(-23+\cosh(2x))\,\mathrm{sech}^4(x)\tanh(x)}{\Gamma(1+2\alpha)} \\
&\quad - \frac{24t^{\alpha+\beta}\,\mathrm{sech}^4(x)\tanh(x)}{\Gamma(1+\alpha+\beta)} + \cdots
\end{aligned} \tag{4.19}
$$

$$
\begin{aligned}
v(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= 1 - 2\tanh^2(x) + \frac{4t^{\beta}\,\mathrm{sech}^2(x)\tanh(x)}{\Gamma(1+\beta)} \\
&\quad + \frac{t^{2\beta}\,\mathrm{sech}^6(x)(21-26\cosh(2x)+\cosh(4x))}{\Gamma(1+2\beta)} \\
&\quad + \frac{48t^{\alpha+\beta}\,\mathrm{sech}^4(x)\tanh^2(x)}{\Gamma(1+\alpha+\beta)} + \cdots
\end{aligned} \tag{4.20}
$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (4.19) becomes

$$
\begin{aligned}
u(x,t) &= \tanh(x) - t\,\mathrm{sech}^2(x) - t^2\,\mathrm{sech}^2(x)\tanh(x) \\
&\quad - \frac{t^3\,\mathrm{sech}^4(x)(-2+\cosh(2x))}{3} + \cdots
\end{aligned} \tag{4.21}
$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (4.20) becomes

$$
\begin{aligned}
v(x,t) &= 1 - 2\tanh^2(x) + 4t\,\mathrm{sech}^2(x)\tanh(x) + 2t^2\,\mathrm{sech}^4(x)(-2+\cosh(2x)) \\
&\quad + \frac{2t^3\,\mathrm{sech}^5(x)(-11\sinh(x)+\sinh(3x))}{3} + \cdots
\end{aligned} \tag{4.22}
$$

The solutions in Eqs. (4.21) and (4.22) are exactly same as the Taylor series expansions of the exact solutions

$$
\begin{aligned}
u(x,t) &= \tanh(x-t) \\
&= \tanh(x) - t\,\mathrm{sech}^2(x) - t^2\,\mathrm{sech}^2(x)\tanh(x) \\
&\quad - \frac{t^3\,\mathrm{sech}^4(x)(-2+\cosh(2x))}{3} + \cdots
\end{aligned} \tag{4.23}
$$

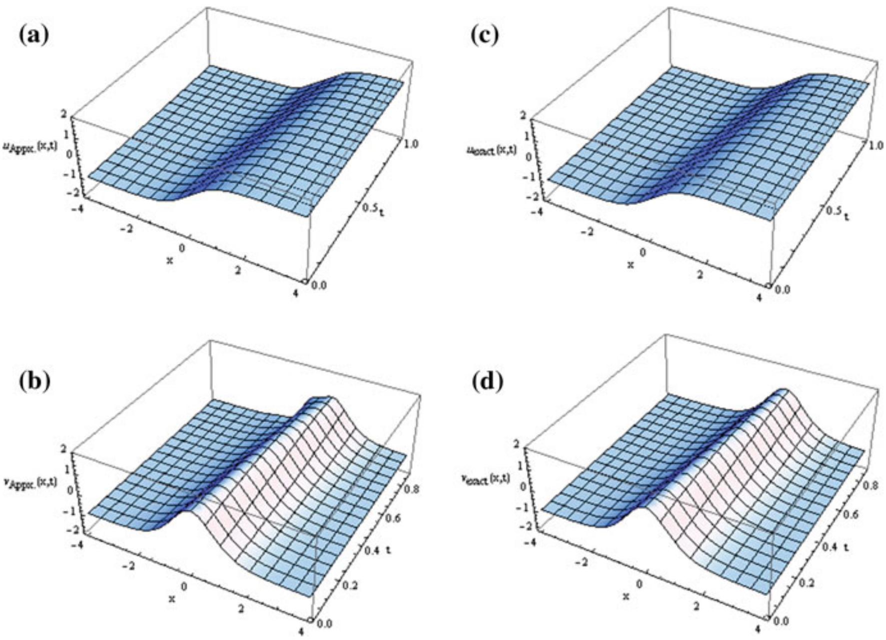

Fig. 2 The surfaces show **a** the numerical approximate solution of $u(x,t)$, **b** the numerical approximate solution of $v(x,t)$, **c** the exact solution of $u(x,t)$, and **d** the exact solution of $v(x,t)$ when $\alpha = 1$ and $\beta = 1$

$$\begin{aligned} v(x,t) &= 1 - 2\tanh^2(x-t) \\ &= 1 - 2\tanh^2(x) + 4t\,\mathrm{sech}^2(x)\tanh(x) + 2t^2\,\mathrm{sech}^4(x)(-2+\cosh(2x) \\ &\quad + \frac{2t^3\,\mathrm{sech}^5(x)(-11\sinh(x)+\sinh(3x))}{3} + \cdots \end{aligned} \tag{4.24}$$

Again, in order to verify the efficiency and reliability of the proposed method for the time fractional coupled modified KdV equations, the graphs have been drawn in Fig. 2a–d. The numerical solutions for Eqs. (4.21) and (4.22) for the special case where $\alpha = 1$ and $\beta = 1$ are shown in Fig. 2a–d. It can be observed from Fig. 2a–d that the soliton solutions obtained by the proposed method are exactly identical with the exact solutions. In this case, we see that the soliton solutions are kink-type for $u(x,t)$ and bell-type for $v(x,t)$.

5 Verification of classical integer order solutions by ADM

In case of $\alpha = 1$ and $\beta = 1$, to solve Eqs. (4.13a) and (4.13b) by means of Adomian decomposition method (ADM), we rewrite the Eqs. (4.13a) and (4.13b) in an operator form

$$L_t u = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3A(u) + \frac{3}{2}\frac{\partial^2 v}{\partial x^2} + 3B(u,v) + 3\frac{\partial u}{\partial x} \tag{5.1}$$

$$L_t v = -\frac{\partial^3 v}{\partial x^3} - 3C(v) - 3G(u,v) + 3H(u,v) - 3\frac{\partial v}{\partial x} \tag{5.2}$$

where $L_t \equiv \frac{\partial}{\partial t}$ is the easily invertible linear differential operator with its inverse operator $L_t^{-1}(.) \equiv \int_0^t (.)d\tau$. Here, the functions $A(u) = u^2\frac{\partial u}{\partial x}$, $B(u,v) = \frac{\partial(uv)}{\partial x}$, $C(v) = v\frac{\partial v}{\partial x}$, $G(u,v) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$ and $H(u,v) = u^2\frac{\partial v}{\partial x}$ are related to the nonlinear terms and they can be expressed in terms of the Adomian polynomials as follows

$A(u) = \sum_{n=0}^{\infty} A_n$, $B(u,v) = \sum_{n=0}^{\infty} B_n$, $C(v) = \sum_{n=0}^{\infty} C_n$, $G(u,v) = \sum_{n=0}^{\infty} G_n$ and $H(u,v) = \sum_{n=0}^{\infty} H_n$. In particular, for nonlinear operator $A(u)$ and $B(u,v)$, the Adomian polynomials are defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[A\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]\Bigg|_{\lambda=0}, n \geq 0,$$

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[B\left(\sum_{k=0}^{\infty}\lambda^k u_k, \sum_{k=0}^{\infty}\lambda^k v_k\right)\right]\Bigg|_{\lambda=0}, n \geq 0,$$

The first few components of $A(u)$, $B(u,v)$, $C(v)$, $G(u,v)$ and $H(u,v)$ are respectively given by

$$\begin{aligned}
A_0 &= u_0^2 u_{0_x}\\
A_1 &= u_0^2 u_{1_x} + 2u_0u_1u_{0_x}\\
A_2 &= u_{0_x}(2u_0u_2 + u_1^2) + u_0^2 u_{2_x} + 2u_0u_1u_{1_x}\\
&\ldots\\
B_0 &= u_0v_{0_x} + v_0u_{0_x}\\
B_1 &= u_0v_{1_x} + v_1u_{0_x} + u_1v_{0_x} + v_0u_{1_x}\\
B_2 &= u_0v_{2_x} + v_2u_{0_x} + u_1v_{1_x} + v_1u_{1_x} + u_2v_{0_x} + v_0u_{2_x}\\
&\ldots\\
C_0 &= v_0v_{0_x}\\
C_1 &= v_0v_{1_x} + v_1v_{0_x}\\
C_2 &= v_1v_{1_x} + v_0v_{2_x} + v_2v_{0_x}\\
&\ldots\\
G_0 &= u_{0_x}v_{0_x}\\
G_1 &= u_{0_x}v_{1_x} + v_{0_x}u_{1_x}\\
G_2 &= u_{1_x}v_{1_x} + v_{0_x}u_{2_x} + u_{0_x}v_{2_x}\\
&\ldots\\
H_0 &= u_0^2 v_{0_x}\\
H_1 &= u_0^2 v_{1_x} + 2u_0u_1v_{0_x}
\end{aligned}$$

$$H_2 = v_{0_x}(2u_0u_2 + u_1^2) + u_0^2 v_{2_x} + 2u_0u_1v_{1_x}$$
$$\ldots$$

and so on, the rest of the polynomials can be constructed in a similar manner.

Now, operating with L_t^{-1} on the both sides of Eqs. (5.1) and (5.2), yields

$$u(x,t) = u(x,0) + L_t^{-1}\left(\frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3A(u) + \frac{3}{2}\frac{\partial^2 v}{\partial x^2} + 3B(u,v) + 3\frac{\partial u}{\partial x}\right) \quad (5.3)$$

$$v(x,t) = v(x,0) + L_t^{-1}\left(-\frac{\partial^3 v}{\partial x^3} - 3C(v) - 3G(u,v) + 3H(u,v) - 3\frac{\partial v}{\partial x}\right) \quad (5.4)$$

The ADM assumes that the two unknown functions $u(x,t)$ and $v(x,t)$ can be expressed by infinite series in the following forms

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (5.5)$$

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) \quad (5.6)$$

Substituting Eqs. (5.5) and (5.6) into Eqs. (5.3) and (5.4) yields

$$u_0(x,t) = u(x,0)$$
$$u_{n+1}(x,t) = L_t^{-1}\left(\frac{1}{2}\frac{\partial^3 u_n(x,t)}{\partial x^3} - 3A_n + \frac{3}{2}\frac{\partial^2 v_n(x,t)}{\partial x^2} + 3B_n + 3\frac{\partial u_n(x,t)}{\partial x}\right), n \geq 0, \quad (5.7)$$

$$v_0(x,t) = v(x,0)$$
$$v_{n+1}(x,t) = L_t^{-1}\left(-\frac{\partial^3 v_n(x,t)}{\partial x^3} - 3C_n - 3G_n + 3H_n - 3\frac{\partial v_n(x,t)}{\partial x}\right), n \geq 0 \quad (5.8)$$

Using known $u_0(x,t)$ and $v_0(x,t)$, all the remaining components $u_n(x,t)$ and $v_n(x,t)$, $n > 0$ can be completely determined such that each terms are determined by using the previous term. From Eqs. (5.7) and (5.8) with Eqs. (4.13c) and (4.13d), we determine the individual components of the decomposition series as

$$u_0 = \tanh(x)$$
$$v_0 = 1 - 2\tanh^2(x)$$
$$u_1 = -t\,\mathrm{sech}^2(x)$$
$$v_1 = 4t\,\mathrm{sech}^2(x)\tanh(x)$$
$$u_2 = -t^2\,\mathrm{sech}^2(x)\tanh(x)$$
$$v_2 = 2t^2(-2 + \cosh(2x))\,\mathrm{sech}^4(x)$$

$$u_3 = -\frac{1}{3}t^3(-2 + \cosh(2x))\,\text{sech}^4(x)$$
$$v_3 = \frac{2}{3}t^3\,\text{sech}^5(x)(-11\sinh(x) + \sinh(3x))$$

and so on, the other components of the decomposition series (5.5) and (5.6) can be determined in a similar way.

Substituting these $u_0, u_1, u_2, \ldots$ and $v_0, v_1, v_2, \ldots$ in Eqs. (5.5) and (5.6), respectively gives the ADM solutions for $u(x, t)$ and $v(x, t)$ in a series form

$$u(x,t) = \tanh(x) - t\,\text{sech}^2(x) - t^2\,\text{sech}^2(x)\tanh(x) - \frac{1}{3}t^3(-2+\cosh(2x))\,\text{sech}^4(x) + \cdots \tag{5.9}$$

$$v(x,t) = 1 - 2\tanh^2(x) + 4t\,\text{sech}^2(x)\tanh(x) + 2t^2(-2+\cosh(2x))\,\text{sech}^4(x) + \frac{2}{3}t^3\,\text{sech}^5(x)(-11\sinh(x)+\sinh(3x)) + \cdots \tag{5.10}$$

Using Taylor series, we obtain the closed form solutions

$$u(x,t) = \tanh(x - t) \tag{5.11}$$
$$v(x,t) = 1 - 2\tanh^2(x - t) \tag{5.12}$$

With initial conditions (4.13c) and (4.13d), the solitary wave solutions of Eqs. (5.1) and (5.2) are of kink-type for $u(x, t)$ and bell-type for $v(x, t)$ which agree to some extent with the results constructed by Fan [9]. According to the learned author Fan [9], the solitary wave solutions of Eqs. (5.1) and (5.2) are kink-type for $u(x, t) = \tanh\left(x + \frac{t}{2}\right)$ and bell-type for $v(x, t) = \frac{3}{2} - 2\tanh^2\left(x + \frac{t}{2}\right)$, where $k = 1$ and $\lambda = -1$. There is definitely a mistake to be reckoned with and should be taken into account for further study. Since using the same parameters $k = 1$ and $\lambda = -1$, the solitary wave solutions of Eqs. (5.1) and (5.2) have been obtained as in Eqs. (5.11) and (5.12).

In the present analysis, the two methods Coupled Fractional Reduced Differential Transform and Adomian decomposition method confirm the justification and correctness of the solutions obtained in Eqs. (5.11) and (5.12).

6 Conclusion

In this paper, a new approximate numerical technique Coupled Fractional Reduced differential transform has been applied for solving nonlinear fractional partial differential equations. The proposed method is only well suited for coupled fractional linear and nonlinear differential equations. In comparison to other analytical methods, the present method is an efficient and simple tool to determine approximate solution of nonlinear coupled fractional partial differential equations. The obtained results demonstrate the reliability of the proposed algorithm and its promising applicability to nonlinear coupled fractional evolution equations. It also exhibits that the proposed method is very efficient and powerful technique in finding the solutions of the nonlinear coupled time fractional differential equations. The main advantage of the proposed method is that it requires less amount of computational overhead in comparison to other numerical and analytical approximate methods and consequently introduces a significant improvement in solving coupled fractional nonlinear equations over existing methods available in open literature. The application of the proposed method for the solutions of time fractional coupled modified KdV equations satisfactorily justifies its simplicity and efficiency. Moreover, in case of integer order coupled modified KdV equations, the obtained results have been verified by the Adomian decomposition method. This investigation leads to the conclusion that soliton solutions for integer order coupled modified KdV equations have been wrongly reported by the reverend author Fan [9].

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